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THE CATEGORY OF Z-CONTINUOUS POSETS

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1. Prologue

Complete distributivity is an old theme in lattice theory: the basic results were already proved in the early fifties by G.N. Raney. Some new features of completely distributive lattices have been discovered recently by P. Dwinger [5] and K.H. Hofmann [9]. The background for [9] is the categorical equivalence between completely distributive lattices and continuous posets, due to R.-E. Hoffmann (see [2, pp. 159-208]) and J.D. Lawson [12]. It is therefore natural to give a presentation of these matters in a more general framework: the category of Z-continuous posets. The study of Z-continuity was suggested by J.B. Wright et al. [14, p. 76]: "... it may be only a curiosity, but we think it would be interesting to investigate this generalized concept ...". The choice of morphisms is not obvious - at least in the non-complete case. Galois connections constitute the main ingredient. For continuous lattices and their generalizations, Galois connections play an important rôle anyway, see [3], [8], [11] and [13]. Now, morphisms between Z-continuous posets can be characterized in terms of pairs of adjoint maps. Instances of this adjunction lemma appear in papers by L. Geissinger and W. Graves [6], K.H. Hofmann and J. Lawson [10], and K.H. Hofmann and A. Stralka [11].

The present paper focusses on the application of Galois connections to continuous posets and their generalizations. Thus we define Z-morphisms (as certain Galois maps), prove the adjunction lemma and show that Z-morphisms preserve various kinds of Z-continuity. Then the Z-version of the following fact is readily obtained: a lattice is continuous if and only if it is the image of an algebraic lattice under a Lawson-continuous map. Moreover, the images of Z-continuous posets under Z-morphisms can be characterized intrinsically. This, for instance, applies to the results in [5].

We assume that the reader is familiar with completely distributive lattices (cf. [1], [7]) and continuous lattices (cf. [7], and [2] for more recent results).

2. Some Z-folklore

In what follows Z always denotes a function which assigns to each poset P a set $Z(P)$ of subsets of P (*Z-sets*) such that all singletons and all isotone images of Z-sets are Z-sets, i.e.

(i) $\{x\} \in Z(P)$ for all $x \in P$, and

(ii) if $\varphi : P \to Q$ is an isotone map of P into some poset Q, then $\varphi(Y) \in Z(Q)$ for all $Y \in Z(P)$.

The lower sets generated by Z-sets are called *Z-ideals* (cf. [14]). As usual

$$\downarrow Y = \{x \in P \mid x \leq y \text{ for some } y \in Y\}$$

denotes the lower set generated by Y. Then

$$I_Z(P) = \{\downarrow Y \mid Y \in Z(P)\}$$

is the system of all Z-ideals of P (ordered by inclusion).

A poset P is called *Z-complete* if every Z-set (or equivalently, every Z-ideal) of P has a supremum in P. A poset P is called *Z-continuous* if P is Z-complete and for each element $y \in P$ there exists a least Z-ideal Y such that $y \leq \bigvee Y$. The *Z-below set* $\Downarrow_Z y$ and the *Z-below relation* $\ll_Z$ are given by

$$x \ll_Z y \;\Leftrightarrow\; x \in \Downarrow_Z y = \bigcap \{Y \in I_Z(P) \mid y \leq \bigvee Y\}.$$

Table 1

Z-set	directed	arbitrary	nonempty finite
Z-ideal	ideal (= directed lower set)	lower set	nonempty finitely generated lower set
Z-complete	up-complete poset	complete lattice	sup-semi-lattice
Z-below	$\ll$ (way below)	$\lll$ (alias Raney's σ)	$\lhd$ (see Section 4)
Z-compact	compact	completely sup-prime	sup-prime
Z-continuous poset	continuous poset	completely distributive lattice	sup-semi-lattice in which every element is a finite sup of sup-primes (see Section 4)
Z-algebraic poset (Z-complete & Z-inductive)	algebraic poset	completely distributive and algebraic lattice	

An element $x \in P$ satisfying $x \ll_Z x$ is called *Z-compact*. The Z-below relation of P is said to be *approximating* if $y = \bigvee \Downarrow_Z y$ for all $y \in P$. Note that a Z-complete poset is Z-continuous if and only if the Z-below relation is approximating and every Z-below set is a Z-ideal. A Z-complete poset P is called *Z-algebraic* if each element $y \in P$ is the supremum of some set K of Z-compact elements such that $\downarrow K$ is a Z-ideal. A poset P is *Z-inductive* (in the sense of [14]) if every element of P is the supremum of some $Y \in Z(C)$ where C is the poset of all Z-compact elements in P. Every Z-complete and Z-inductive poset is Z-algebraic, and for most of the relevant choices of Z, the converse is also true. However, one counterexample is obtained by taking for $Z(P)$ the system of all upper bounded subsets of P. In this case the complete chain $\omega + 1 = \{0, 1, 2, \dots, \omega\}$ is an example of a Z-algebraic poset which is not Z-inductive because there is no set of Z-compact (i.e. completely sup-prime) elements which has both a Z-compact upper bound and the supremum ω.

The standard examples of Z are given by selecting directed sets, arbitrary sets, and (nonempty) finite sets, respectively. See Table 1.

Of course, the Z-below relation is always transitive. In each of the preceding examples, it is even idempotent on any Z-continuous poset. In the context of continuous lattices this property of the way-below relation is usually referred to as the *interpolation property*. For arbitrary Z, we call a Z-continuous poset P *strongly Z-continuous* if it admits interpolation, that is, $x \ll_Z y$ implies that $x \ll_Z w \ll_Z y$ for some $w \in P$ (cf. [3], [12]). Note that *every Z-algebraic poset is strongly Z-continuous*. Indeed, a poset is Z-algebraic if and only if it is Z-continuous and

$$x \ll_Z y \Leftrightarrow x \le w \le y \quad \text{for some Z-compact element } w.$$

Unfortunately, we do not have any example of a Z-continuous poset missing the interpolation property. For the usual choices of Z the notions of 'Z-continuous' and 'strongly Z-continuous' always coincide. This is due to the fact that in most instances the function Z under consideration is union-complete (in the sense of [14]): Z is called *union-complete* if for every poset P, the poset $I_Z(P)$ is closed under Z-unions, that is, the union of any Z-set of $I_Z(P)$ is a Z-ideal of P. As in the case of continuous posets, one can prove the following: *If Z is union-complete, then every Z-continuous poset is strongly Z-continuous* (see the Theorem below). However, the assumption of union-completeness is not always necessary for this conclusion. If, for example, Z selects all subsets with at most two elements then every Z-continuous poset is already Z-algebraic (and therefore strongly Z-continuous) although Z is not union-complete.

In [14] it has been shown that *for every union-complete Z, the Z-complete Z-inductive posets are, up to isomorphism, just the Z-ideal systems $I_Z(P)$; in particular, $I_Z(P)$ is Z-algebraic. The compact elements of $I_Z(P)$ are precisely principal ideals of P.*

Let $\varphi : P \to Q$ and $\psi : Q \to P$ be maps between posets P and Q such that

$$\psi x \le v \quad \text{if and only if} \quad x \le \varphi v \qquad (v \in P, x \in Q).$$

Then ψ is called a *lower adjoint* of φ, and φ is called an *upper adjoint* of ψ. Adjoint maps determine each other uniquely. Maps having a lower adjoint are known in the literature under various names: right adjoint, left adjoint, residual, Galois maps, etc. The reader will find the necessary information in the books [4] and [7]. The definition of Z-continuity can be rephrased in terms of adjoint maps: *a Z-complete poset is Z-continuous if and only if the supremum map*

$$\vee : I_Z(P) \to P, \quad Y \mapsto \bigvee Y$$

has a lower adjoint. The lower adjoint is then, of course, the Z-below map

$$\Downarrow_Z : P \to I_Z(P), \quad y \mapsto \Downarrow_Z y \quad \text{(cf. [13])}.$$

3. Morphisms versus comorphisms

Henceforth we will briefly write $\ll$ and $\Downarrow$ instead of $\ll_Z$ and $\Downarrow_Z$.

A map $\varphi : P \to Q$ of a Z-complete poset P into a poset Q is called a *Z-morphism* if φ has a lower adjoint and preserves Z-sups:

$$\varphi(\bigvee Y) = \bigvee \varphi(Y) \quad \text{for all } Y \in Z(P).$$

A map $\psi : Q \to P$ is called a *Z-comorphism* if ψ has an upper adjoint and preserves the Z-below relation:

$$x \ll y \text{ in } Q \quad \text{implies} \quad \psi x \ll \psi y \text{ in } P.$$

For Z-algebraic posets Q the latter is equivalent to the condition that ψ maps Z-compact elements onto Z-compact elements. Morphisms and comorphisms correspond to each other as one expects: for any adjoint pair (φ, ψ) of maps between Z-continuous posets, φ is a Z-morphism if and only if ψ is a Z-comorphism. In fact, we have the following basic lemma.

Lemma. *Let $\varphi : P \to Q$ be a map between Z-complete posets P and Q which has a lower adjoint ψ. If φ is a Z-morphism then ψ is a Z-comorphism. If the Z-below relation of Q is approximating and ψ is a Z-comorphism, then φ is a Z-morphism.*

Proof. Assume that φ preserves Z-sups. Let $x \ll y$ in Q. If $\psi y \le \bigvee U$ for some $U \in Z(P)$, then $\varphi(U)$ is a Z-set in Q such that $y \le \varphi(\bigvee U) = \bigvee \varphi(U)$, whence there exists $u \in U$ with $x \le \varphi u$, that is, $\psi x \le u$ Therefore $\psi x \ll \psi y$ in P, and thus ψ is a Z-comorphism.

Conversely, assume that ψ preserves $\ll$. Let U be any Z-set of P, and put $y = \varphi(\bigvee U)$. Then $\psi y = \psi\varphi(\bigvee U) \le \bigvee U$. Since ψ preserves $\ll$, we obtain

$$\psi(\Downarrow y) \subseteq \Downarrow \psi y \subseteq \Downarrow \bigvee U \subseteq \downarrow U,$$

whence $\Downarrow y \subseteq \downarrow \varphi(U)$. Therefore, if $\ll$ is approximating on Q then

$$\varphi(\bigvee U) = y = \bigvee \Downarrow y \le \bigvee \varphi(U).$$

Consequently, φ preserves Z-sups. $\square$

Let $\mathcal{ZCP}$ denote the category of Z-continuous posets and Z-morphisms. The Lemma provides a concrete representation of the category opposite to $\mathcal{ZCP}$. Our main result confirms that we have chosen the right morphisms.

Theorem. *The image of a Z-continuous (strongly Z-continuous, resp.) poset under a Z-morphism is Z-continuous (strongly Z-continuous, resp.). For a union-complete function Z, a poset is (strongly) Z-continuous if and only if it is the image of a Z-algebraic poset under a Z-morphism.*

Proof. Let $\varphi : P \to Q$ be a Z-morphism of a Z-continuous poset P onto a poset Q. Then the lower adjoint $\psi : Q \to P$ of φ is one-to-one, and the composition $\varphi\psi$ is the identity map on Q. The proof now proceeds in a number of steps.

(1) φ maps Z-ideals onto Z-ideals.

For, let U be a Z-set of P. Then $\varphi(\downarrow U) \subseteq \downarrow\varphi(U)$ because φ is isotone. Since φ has a lower adjoint ψ, $x \in \downarrow\varphi(U)$ implies that $\psi x \in \downarrow U$ and hence $x = \varphi\psi x \in \varphi(\downarrow U)$. Therefore $\varphi(\downarrow U)$ is the lower set generated by the Z-set $\varphi(U)$.

(2) Q is Z-complete.

Let Y be a Z-set of Q. Then $\psi(Y)$ is a Z-set of P and so $\bigvee \psi(Y)$ exists. Certainly, $\varphi(\bigvee\psi(Y))$ is an upper bound of $\varphi\psi(Y) = Y$. On the other hand, for any upper bound z of Y we get $\varphi(\bigvee\psi(Y)) \leq \varphi\psi z = z$. Therefore $\varphi(\bigvee\psi(Y))$ is the supremum of Y.

(3) $u \ll \psi y$ in P implies $\varphi u \ll y$ in Q.

Let $y \leq \bigvee Y$ for some Z-set Y of Q. Then by (2), $y \leq \varphi(\bigvee\psi(Y))$, and so $\psi y \leq \bigvee\psi(Y)$. As $\psi(Y)$ is a Z-set in P, we get $\varphi(\Downarrow\psi y) \subseteq \varphi(\downarrow\psi(Y)) \subseteq \downarrow\varphi\psi(Y) = \downarrow Y$. Thus $\varphi(\Downarrow\psi y) \subseteq \Downarrow y$.

(4) $\Downarrow y = \varphi(\Downarrow\psi y) \in I_Z(Q)$ for $y \in Q$.

We have seen in (3) that $\varphi(\Downarrow\psi y)$ is contained in $\Downarrow y$. If $x \ll y$, then $\psi x \ll \psi y$ by the Lemma, and hence $x = \varphi\psi x \in \varphi(\Downarrow\psi y)$. This proves the required equality, and therefore $\Downarrow y$ is a Z-ideal by (1).

(5) $\ll$ is approximating on Q.

Indeed, since $\ll$ is approximating on P and φ preserves Z-sups, we deduce from (4) that

$$y = \varphi\psi y = \varphi(\bigvee\Downarrow\psi y) = \bigvee\varphi(\Downarrow\psi y) = \bigvee\Downarrow y.$$

From (2), (4), and (5) we infer that Q is a Z-continuous poset.

(6) $x \ll y$ in Q if and only if $\psi x \ll \psi y$ in P.

Indeed, $\psi x \ll \psi y$ implies $x = \varphi\psi x \ll y$ by (3).

(7) $\quad \Downarrow \psi y = \downarrow \psi(\Downarrow y)$ for $y \in Q$.

The Z-below set $\Downarrow y$ is a Z-ideal, and hence so is the lower set $\downarrow \psi(\Downarrow y)$. The supremum of the latter is $\psi(\bigvee \Downarrow y)$, and $\psi(\bigvee \Downarrow y) = \psi y$ by (5) (because a lower adjoint map preserves all existing suprema). Since $\Downarrow \psi y$ is the least Z-ideal whose supremum is ψy, the Z-ideal $\downarrow \psi(\Downarrow y)$ must contain $\Downarrow \psi y$. The converse inclusion follows from the fact that ψ preserves the Z-below relation.

(8) $\quad$ If P admits interpolation, then so does Q.

Let $x \ll y$ in Q. Then $\psi x \ll \psi y$, and thus there exists $v \in P$ with $\psi x \ll v \ll \psi y$ by interpolation in P. In view of (7) there exists $w \ll y$ such that $v \leq \psi w$. Hence by (6) we get $x \ll w \ll y$, as desired.

We conclude that Q is strongly Z-continuous whenever P is strongly Z-continuous.

(9) $\quad$ If Z is union-complete, then $I_Z(P)$ is a Z-algebraic poset, and the supremum map $v : I_Z(P) \to P$ is a Z-morphism.

In fact, the map v preserves Z-sups, and by Z-continuity of P, v has a lower adjoint.

Now, by (9) and what has been shown before, the proof of the Theorem is complete. □

With the Lemma in hand, we can reformulate the last part of the Theorem in a dualized version. *For union-complete Z, every Z-continuous poset P can be embedded in a Z-algebraic poset* (*viz.* $I_Z(P)$) *by a Z-comorphism* (*viz. the Z-below map*). Notice, however, that a poset which is embedded in a Z-algebraic poset by a Z-comorphism need not be Z-continuous.

Now suppose that P is an arbitrary poset and $\psi : Q \to P$ is some one-to-one map having an upper adjoint φ. Then $\psi(Q) = \psi\varphi(P)$ is an isomorphic copy of Q. The map $\psi\varphi$ is a kernel map of P: a *kernel map* $\kappa : P \to P$ is an isotone map such that $\kappa\kappa u = \kappa u \leq u$ for all $u \in P$. A subset K of P is a *kernel subset* if K is the image $\operatorname{im} \kappa = \kappa(P)$ of some kernel map κ. Recall that a kernel subset is closed under arbitrary (existing) suprema. Now, if $\kappa : P \to P$ is any kernel map, then the inclusion map $\iota : \operatorname{im} \kappa \to P$ of the kernel subset is a lower adjoint of the onto map $\kappa : P \to \operatorname{im} \kappa$. Hence by the Lemma, if P is Z-complete and κ preserves Z-sups, then ι preserves $\ll$ and thus is a Z-comorphism. Conversely, if ι preserves $\ll$, then κ preserves Z-sups provided that $\ll$ is approximating on $\operatorname{im} \kappa$. In this case, κ can be regarded as a Z-morphism of P onto $\operatorname{im} \kappa$; further by (6), the Z-below relation of $\operatorname{im} \kappa$ is just the restriction of the Z-below relation of P. Finally observe that for any Z-morphism $\varphi : P \to Q$ the corresponding kernel map $\kappa = \psi\varphi$ (where ψ is the lower adjoint of φ) preserves Z-sups. *Hence every image of a Z-continuous poset P under a Z-morphism can be represented as a kernel subset of P whose Z-below relation is the restriction of the Z-below relation of P. Conversely, every Z-continuous* *kernel subset of P with the latter property is the image of P under a Z-morphism. For union-complete Z, every Z-continuous poset Q may be regarded as a kernel subset of some Z-algebraic poset P such that* $x \lll y$ *in Q implies* $x \lll y$ *in P.*

Implicit in the foregoing is an intrinsic description of the images of a Z-continuous poset under Z-morphisms. Let P be a Z-complete poset. An equivalence relation Θ on P is called a *Z-congruence* if Θ is the kernel of some Z-morphism φ of P onto a poset Q, that is, $u\,\Theta\,v$ if and only if $\varphi u = \varphi v$. Then Θ is also the kernel of the map $\psi\varphi$, where ψ is the lower adjoint of φ. In particular, $\psi\varphi v$ is the least element of the Θ-class containing $v \in P$. In view of the above observations the following characterization of Z-congruences is now evident: *for any Z-complete poset P, Z-congruences can be identified with kernel maps preserving Z-sups. If, in addition, P is Z-continuous, then there is a one-to-one correspondence between Z-congruences and Z-continuous kernel subsets of P whose Z-below relations are restrictions of the Z-below relation of P.*

4. Epilogue

As was mentioned before, instances of $\mathcal{ZCP}$ are (i) the category $\mathcal{CP}$ of continuous posets and (ii) the category $\mathcal{CD}$ of completely distributive lattices. For the full subcategory $\mathcal{CL}$ of $\mathcal{CP}$, that is, for continuous lattices, the results of Section 3 belong to the folklore, cf. Chapters I.2, I.4, and IV.1 of the compendium [7]. Note that a map between complete lattices is a Z-morphism if and only if it preserves arbitrary infs and Z-sups. Then a Z-congruence is an equivalence relation compatible with arbitrary infs and Z-sups. Hence complete congruences on complete lattices can be identified with kernel maps having an upper adjoint. This fact is well known, see [4, Theorem 15.1]. Moreover, Dwinger's main results in [5] (viz. Theorems 3.4 and 4.3) follow from our observations in Section 3: for instance, the lattice of complete congruences of a completely distributive lattice L is antiisomorphic to the lattice of completely distributive kernel subsets K of L such that $x \lll y$ in K implies $x \lll y$ in L.

There are, of course, further instances of, say, the Lemma. For each poset let Z select all nonempty finite subsets. Then a Z-complete poset P is just a sup-semilattice. Let $\lhd$ denote its Z-below relation. The Z-compact elements $x \lhd x$ are those elements which are sup-prime. The relation $\lhd$ is certainly approximating on P whenever P is generated by sup-primes (that is, every element of P is a supremum of sup-prime elements). Let $\varphi : P \to Q$ be a map between sup-semilattices generated by sup-primes which has a lower adjoint ψ. Then φ preserves finite suprema if and only if ψ maps sup-prime elements onto sup-prime elements. The order-theoretic dual of this statement is basic to the spectral theory of continuous Heyting algebras, see K.H. Hofmann and J. Lawson [10]. Notice that even a complete lattice generated by sup-primes (i.e., the lattice of closed sets in a topological space) need not be Z-algebraic. Indeed, for the present function Z, either of the properties 'Z-

algebraic' and 'Z-continuous' means "sup-semilattice in which every element is a finite supremum of sup-primes", because in a Z-continuous poset the finitely many maximal elements of any Z-below set must be Z-compact, that is, sup-prime.

The reader will have no difficulties in finding more applications of the manipulation rules for the general Z.

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